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Constraint quantization of a worldline system invariant under reciprocal relativity: II

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Abstract

We consider the worldline quantization of a system invariant under the symmetries of reciprocal relativity. Imposition of the first class constraint, the generator of local time reparametrizations, on physical states enforces identification of the worldline cosmological constant with a fixed value of the quadratic Casimir of the quaplectic symmetry group $Q(3,1) \cong U(3,1) \ltimes H(4)$, the semi-direct product of the pseudo-unitary group with the Weyl–Heisenberg group. In our previous paper, *J. Phys. A: Math. Theor.* **40** (2007) 12095, the 'spin' degrees of freedom were handled as covariant oscillators, leading to a unique choice of cosmological constant, required for projecting out negative-norm states from the physical gauge-invariant states. In the present paper, the spin degrees of freedom are treated as standard oscillators with positive norm states (wherein Lorentz boosts are not number-conserving in the auxiliary space; reciprocal transformations are of course not spin-conserving in general). As in the covariant approach, the spectrum of the square of the energy–momentum vector is continuous over the entire real line, and thus includes tachyonic (spacelike) and null branches. Adopting standard frames, the Wigner method on each branch is implemented, to decompose the auxiliary space into unitary irreducible representations of the respective little algebras and additional degeneracy algebras. The physical state space is vastly enriched as compared with the covariant approach, and contains towers of integer spin massive states, as well as unconventional massless representations of continuous spin type, with continuous Euclidean momentum and arbitrary integer helicity.

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1. Introduction

In our previous paper [1] (hereinafter referred to as I), we initiated an investigation into worldline formulations of elementary systems possessing the extended symmetries of reciprocal relativity. These make up the so-called quaplectic[3] group $Q(D-1,1) \cong U(D-1,1) \ltimes H(D)$, the semi-direct product of the pseudo-unitary group with the Weyl–Heisenberg group in D dimensions. The latter is realized as the central extension of a $(2D+1)$-dimensional translation group, in accord with the original vision of Born [2–5] whereby 'position' and 'momentum' are reciprocally equivalent, and generalized in recent work by Low [6–8] to encompass the full $(D+1)^2$-dimensional quaplectic transformation group. The novel feature in our formulation was a compact 'phase' coordinate $\theta(\tau)$, in addition to worldline 'position' and 'momentum' coordinates $x^\mu(\tau)$ and $p_\mu(\tau)$; the conserved θ-momentum thus introduces a quantum number with a discrete spectrum and sets the scale of Planck's constant in the Heisenberg algebra (and can be regarded as a superselected quantity).

The worldline model presented in I is defined by the action (in D-dimensional Minkowski space)

$$S = \int \mathrm{d}\tau \left(\frac{1}{e(\tau)} L + \Lambda e(\tau) \right),$$
$$L = -\frac{1}{2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau} + \kappa_0^2 \frac{\mathrm{d}p_\nu}{\mathrm{d}\tau}\frac{\mathrm{d}p^\nu}{\mathrm{d}\tau}\right) + \mathtt{C}\frac{\kappa_0}{\lambda_0}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau} - \lambda_0 \left(x^\mu(\tau)\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} - p_\nu(\tau)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right)\right)^2, \tag{1}$$

where $e(\tau)$ is the worldline einbein, $\kappa_0 = c/b$, where b is Low's parameter of maximum force (rate of change of momentum), c is the speed of light, λ_0 has units of action^{-1}, and $\mathtt{C}$ is an arbitrary numerical constant. In I, a complete analysis of Noether charges and constraints arising from the symmetries of (1) was presented (for the methodology of constraint quantization see, for example, [9]). The state space of the system (before imposition of the constraint) corresponds to the direct product of *two* independent D-dimensional Heisenberg algebras—one generated by the conserved generators $\mathcal{X}^\mu, \mathcal{P}_\nu$ of translations in the original 'position' and 'momentum' worldline coordinates, and a second auxiliary, non-conserved set $\mathbb{X}^\mu, \mathbb{P}_\nu$ (both sets have the same central extension, the conserved θ-momentum, Π_θ). Imposition of the first class constraint, the generator of local time reparametrizations, on physical state space enforces identification [10] of the worldline cosmological constant Λ in the model, with a fixed value of the quadratic Casimir of the quaplectic symmetry group. However, both sets of Heisenberg generators provide the material for the construction of the (conserved) generators of the homogeneous $U(D-1,1)$ component of the quaplectic algebra,

$$E_{\mu\nu} := \tfrac{1}{2}\{\overline{\mathcal{Z}}_\mu, \mathcal{Z}_\nu\} + \tfrac{1}{2}\{\overline{\mathbb{Z}}_\mu, \mathbb{Z}_\nu\} \tag{2}$$

in a complex basis with covariant combinations

$$\mathcal{Z}_\mu = \tfrac{1}{\sqrt{2}}(-\mathcal{X}_\mu + \mathrm{i}\mathcal{P}_\nu), \qquad \mathbb{Z}_\mu = \tfrac{1}{\sqrt{2}}(-\mathbb{X}_\mu + \mathrm{i}\mathbb{P}_\nu), \tag{3}$$

and wherein the generators of Lorentz transformations are, for example, the total orbital-type combinations $(\mathcal{X}_\mu\mathcal{P}_\nu - \mathcal{X}_\nu\mathcal{P}_\mu) + (\mathbb{X}_\mu\mathbb{P}_\nu - \mathbb{X}_\nu\mathbb{P}_\mu)$. Defining $U = E^\mu{}_\mu$, the $Q(D-1,1)$ Casimir operator reads

$$C_1 = U\Pi_\theta - \tfrac{1}{2}(\mathcal{P}^\mu\mathcal{P}_\mu + \mathcal{X}^\mu\mathcal{X}_\mu) \equiv \tfrac{1}{2}(\mathbb{P}^\mu\mathbb{P}_\mu + \mathbb{X}^\mu\mathbb{X}_\mu), \tag{4}$$

[3] See S G Low, e-print arXiv:math-ph/0502018 for an explanation of this name.

which is immediately recognizable as an ingredient of the first class constraint, to be imposed on physical state space as an operator condition,

$$\tfrac{1}{2}(\mathbb{P}^{\mu}\mathbb{P}_{\mu} + \mathbb{X}^{\mu}\mathbb{X}_{\mu}) + \tfrac{1}{2}\Pi_{\theta}^{2} = \Lambda. \tag{5}$$

Thus for a nontrivial solution, the worldline cosmological constant must lie in the spectrum of the quaplectic Casimir, modulo the square of the conserved θ-momentum[4].

In I, the auxiliary operators $\mathbb{X}^{\mu}$, $\mathbb{P}_{\nu}$ were treated as 'covariant oscillators', with a Lorentz invariant zero-mode state, but acting on a space containing some negative-norm states, associated with the fact that the indefinite sign of the Minkowski metric necessarily leads to at least one set of raising and lowering operators with a 'wrong-sign' commutation relation. In this case the only consistent solution is to arrange things so that the cosmological constant balances the contribution to the constraint from Π_{θ}, and only the vacuum state itself survives; the auxiliary space collapses, and only spinless fields occur in the spectrum. However, because there are no other restrictions on the physical energy–momentum vector $\mathcal{P}_{\nu}$, its eigenvalues p_{ν} will produce a continuous range of values of $p \cdot p$, including not only massive and massless, but also unphysical[5], tachyonic (spacelike) branches. We refer the reader to I for a full discussion of the model.

In the present paper, the auxiliary space is treated using standard oscillators, with correct-sign commutation relations. As a result, only positive-norm states occur (the auxiliary space is therefore isomorphic to a product of standard $L^{2}(\mathbb{R})$ spaces). Compared with the covariant approach however, the zero-mode state in the auxiliary space is not Lorentz- or Born reciprocal-invariant, but takes its place in the weight decompositions of the appropriate extended little algebras. The auxiliary 'spin'-containing space is vastly enriched as compared with the covariant approach, and notwithstanding the continuous range of $p \cdot p$, it is instructive to carry out a careful case-by-case analysis of the spin content.

It is this task which we are at pains to present in detail here, with the restriction to $D = 4$ dimensions, so that the full spectrum of states in the physical state space after imposition of the constraint, with both physical and 'unphysical' particle contents, can be compared with the standard classification of unitary irreducible representations of the Poincaré group [11].

As already mentioned, the physical worldline model defined by (1) is developed in I, to which the reader is referred for details. In section 2, we commence analysis of the problem of identifying the appropriate little algebras, and accompanying degeneracy algebras, for each of the orbit classes $p \cdot p > 0, < 0, \equiv 0$ and $= 0$ (massive, spacelike (tachyonic), null and massless, respectively). We take as read the details of the quantization of the model carried out in I; for completeness, however, we include in appendix A.1 a discussion of the structure of the quaplectic Lie algebra and bases relevant for physics. The main technical details of the derivation of group branching rules are held over to appendix A.2 (unitary irreducible representations of $Sp(2, \mathbb{R})$ and products of discrete series representations), appendix A.3 (dual subalgebras of $Sp(8, \mathbb{R})$ and the $> 0, < 0, \equiv 0$ cases), appendix A.4 (massive case), appendix A.5 (tachyonic, spacelike case), appendix A.6 (null case), and finally appendix A.7 (dual $E(2)$ subalgebras and massless states).

Results are collected in summary form as tables 1 and 2. The analysis of representations found is carried out in section 3, with attention to how the constraint selects the physical state

[4] See I for details of the correct way to scale out physical constants to produce the dimensionless degrees of freedom assumed in the present treatment. For example, the correct definition of the Heisenberg algebra generators depends on the sign of the Π_{θ} eigenvalue (assuming this to be nonvanishing).

[5] In the following, a distinction is intended between 'physical state space', meaning the state space after imposition of the constraint, and 'physical and unphysical particles', meaning that, as well as standard particle states, physical state space also contains unconventional, and hence unphysical in that sense, particle content.

Table 1. Standard frame momentum coordinates, little algebras and duals, $\mathcal{L} \times \mathbb{L}$, for $p \cdot p > 0$ (massive), <0 (tachyonic), $p \equiv 0$ (null) and $= 0$ (massless), together with the constraint operator to be projected onto eigenvalue Δ, expressed in terms of the diagonal generators of the dual algebras (for discussion and notation see section 3 and appendix A).

	$\mathring{p}$	$\mathcal{L}$	$\mathbb{L}$	$\frac{1}{2}(\widehat{N}_1 + \widehat{N}_2 + \widehat{N}_3 - \widehat{N}_0) + \frac{1}{2}$
>0	$(1, 0, 0, 0)$	$SO(3)$	$Sp^{(0)}(2, \mathbb{R}) + Sp^{(123)}(2, \mathbb{R})$	$K_0^{(0)} + K_0^{(123)}$
<0	$(0, 0, 0, 1)$	$SO(3)$	$Sp^{(012)}(2, \mathbb{R}) + Sp^{(3)}(2, \mathbb{R})$	$K_0^{(012)} + K_0^{(3)}$
$\equiv 0$	$(0, 0, 0, 0)$	$SO(3, 1)$	$Sp^{(0123)}(2, \mathbb{R})$	$K_0^{(0123)}$
$=0$	$(1, 0, 0, 1)$	$\mathcal{E}(2)$	$\mathbb{E}(2)$	$\mathbb{M}$

Table 2. Branches of the mass-squared spectrum of physical states, $p \cdot p > 0$ (massive), $p \equiv 0$ (null), <0 (tachyonic) and $= 0$ (massless) together with irreducible representations of the respective little algebras and associated degeneracy algebras, to be projected with respect to the constraint eigenvalue Δ (for discussion and notation see table 1, section 3 and appendix A).

$p \cdot p$	$\mathcal{L}$	$\mathbb{L}$		
>0	$SO(3)$	$Sp^{(123)}(2, \mathbb{R})$	$Sp^{(0)}(2, \mathbb{R})$	
	$[\ell]$	$D^+_{-\frac{1}{2}\ell-\frac{3}{4}}$	$D^-_{-\frac{1}{4}} \oplus D^-_{-\frac{3}{4}}$	$\sum_{\ell=0}^{\infty}$
<0	$SO(2, 1)$	$Sp^{(012)}(2, \mathbb{R})$	$Sp^{(3)}(2, \mathbb{R})$	
	$[0]$	$D^+_{-\frac{1}{4}} \oplus D^-_{-\frac{1}{4}}$	$D^+_{-\frac{1}{4}} \oplus D^+_{-\frac{3}{4}}$	
	D^+_ℓ	$D^+_{-\frac{1}{2}\ell+\frac{1}{4}}$	$D^+_{-\frac{1}{4}} \oplus D^+_{-\frac{3}{4}}$	$\sum_{\ell=1}^{\infty}$
	D^-_ℓ	$D^+_{-\frac{1}{2}\ell+\frac{1}{4}}$	$D^+_{-\frac{1}{4}} \oplus D^+_{-\frac{3}{4}}$	$\sum_{\ell=1}^{\infty}$
	$D\left(-\frac{1}{2} + is\right)$	$2D\left(-\frac{1}{2} + 2is\right)$	$D^+_{-\frac{1}{4}} \oplus D^+_{-\frac{3}{4}}$	$\int_0^\infty \mathrm{d}s$
$\equiv 0$	$SO(3, 1)$	$Sp^{(0123)}(2, \mathbb{R})$		
	$D[\ell, 0]$	$D^+_{-\frac{1}{2}\ell-\frac{1}{2}}$		$\sum_{\ell=0}^{\infty}$
	$D[0, -2is]$	$2D\left(-\frac{1}{2} + is\right)$		$\int_0^\infty \mathrm{d}s$
$=0$	$\mathcal{E}(2)$	$\mathbb{E}(2)$		
	$D(\pi\pi_\parallel)$	$D(\pi_\parallel)$		$\int\int_0^\infty \mathrm{d}\pi\, \mathrm{d}\pi_\parallel$

space in each case. The overview of results is complemented by a discussion of potential future directions and refinements.

2. The Wigner method for the extended worldline system

As must be the case for valid quantization of any system admitting classical symmetries, the corresponding state space carries appropriate unitary representations of the symmetry group in question. In the case of the worldline system (1), this is of course the quaplectic Lie group $Q(3, 1) \cong U(3, 1) \ltimes H(4)$.

The structure of the corresponding Lie algebra is given in appendix A.1. As is clear from (2) above, the homogeneous generators $E^\mu{}_\nu$ of $U(3, 1)$ are constructed using the material provided by both independent four-dimensional Heisenberg algebras (generated by $\mathcal{Z}_\mu, \overline{\mathcal{Z}}_\nu$, as well as the auxiliary set $\mathbb{Z}_\mu, \overline{\mathbb{Z}}_\nu$); whereas the *physical* Weyl–Heisenberg group is generated by the set $\mathcal{Z}_\mu, \overline{\mathcal{Z}}_\nu$, associated with the conserved Noether charges. However, from (4) it turns out that the $Q(3, 1)$ Casimir invariant C_1 depends only on the auxiliary Heisenberg generators.

In fact from (2) and the construction (A.8), it is clear that the derived quantities $e^{\mu}{}_{\nu}$, which are generators of $U(3,1)$ but which *commute* with the physical Heisenberg algebra, are precisely $\frac{1}{2}\{\overline{\mathbb{Z}}^{\mu}, \mathbb{Z}_{\nu}\}$; the $Q(3,1)$ Casimir invariants are traces of matrix products of $e^{\mu}{}_{\nu}$ (see appendix A.1) and in this realization of the $U(3,1)$ Lie algebra, it is known [12] that the linear one is the only independent Casimir invariant. The role of the auxiliary space carrying representations of $U(3,1) \supset O(3,1)$ via the $e^{\mu}{}_{\nu}$ is brought out for instance by the expression for the generators of Lorentz transformations (see (2), (A.12)), namely

$$L_{\mu\nu} = \mathrm{i}(\mathcal{X}_{\mu}\mathcal{P}_{\nu} - \mathcal{X}_{\nu}\mathcal{P}_{\mu}) + \mathrm{i}(\mathbb{X}_{\mu}\mathbb{P}_{\nu} - \mathbb{X}_{\nu}\mathbb{P}_{\mu}) \equiv \mathrm{i}(\mathcal{X}_{\mu}\mathcal{P}_{\nu} - \mathcal{X}_{\nu}\mathcal{P}_{\mu}) + \mathbb{L}_{\mu\nu} \qquad (6)$$

which means that $\mathbb{L}_{\mu\nu}$ is identified with spin.

With a view to its reduction with respect to unitary irreducible representations of the spacetime Poincaré algebra, it is possible to diagonalize the physical 4-momentum $\mathcal{P}_{\mu}$ (adopting the usual quantum-mechanical Schrödinger representation with $\mathcal{P}_{\mu} \to -\mathrm{i}\hbar\partial/\partial x^{\mu}$ for suitable functions on Minkowski space), but for the auxiliary space, for enumerative purposes, to identify the auxiliary four-dimensional Heisenberg algebra with four pairs of standard oscillator raising and lowering operators, via (A.13). With the standard commutation relations

$$[a, a^{\dagger}] = 1, \qquad \left[b_i, b_j^{\dagger}\right] = \delta_{ij}, \qquad i, j = 1, 2, 3,$$

and the usual introduction of the zero-mode state annihilated by the lowering operators, a suitable basis for the auxiliary space is therefore provided by the number states $|n_0, n_1, n_2, n_3\rangle$ (see (A.14)), $n_0, n_i = 0, 1, \ldots$ being the eigenvalues of the respective number operators $\widehat{N}_0 = a^{\dagger}a$, $\widehat{N}_i = b_i^{\dagger}b_i$, $i = 1, 2, 3$.

Finally the full quantum space of states (before imposition of the constraint) is spanned by the basis[6]

$$|p^{\mu}\rangle \otimes |n_0, n_1, n_2, n_3\rangle \equiv |p^{\mu}; n_0, n_1, n_2, n_3\rangle. \qquad (7)$$

Now, from the general expression for the Casimir invariant (A.11), and its form (4) in the present realization, the constraint (5) to be imposed on physical states amounts to the projection of the generic number states $|p^{\mu}; n_0, n_1, n_2, n_3\rangle$ onto a certain selected eigenspace of the covariant oscillator Hamiltonian, $\frac{1}{2}(\mathbb{P}\cdot\mathbb{P} + \mathbb{X}\cdot\mathbb{X})$. For consistency with the identifications to be made below between this operator and various key symmetry generators, we adopt the following form of the constraint:

$$\tfrac{1}{2}(n_1 + n_2 + n_3 - n_0) + \tfrac{1}{2} = \Delta, \qquad \text{where} \quad \Delta := \tfrac{1}{2}\Lambda - \tfrac{1}{4}(n_\theta + \sigma)^2, \qquad (8)$$

where we assume that a fixed eigenvalue $n_\theta + \sigma$ of the compact θ-momentum Π_θ has been selected, for some integral n_θ, up to a modular parameter $0 \leqslant \sigma < 1$ (see [13] and I).

Since $p^{\mu} \in \mathbb{R}^4$, there is *no* restriction on the energy–momentum squared, $p\cdot p$. In analyzing the particle content of physical state space by carrying out the complete decomposition of the (irreducible) unitary representations of $Q(3,1)$ specified by the states (7) subject to (8), with respect to the Poincaré group, the spectrum in the massive and tachyonic (spacelike) branches will therefore be a continuum, with $\infty > p\cdot p > 0$, and $-\infty < p\cdot p < 0$ respectively. Moreover, we will see below that, for both the massless and null cases where $p\cdot p \equiv 0$, the reduction also turns out to be in the form of a direct integral.

From (6), it is necessary to refine the physical state space analysis by diagonalizing the *second* Poincaré group Casimir, corresponding to the spin quantum number. For this task we take up the traditional method of Wigner [11]: for each orbit class, it is sufficient to work with

[6] A further direct product with a one-dimensional space associated with the fixed θ-momentum has been omitted for the sake of clarity; see (8) below.

4-momentum fixed in some standard frame, $\overset{\circ}{p}$, and give the complete reduction of state space with respect to the appropriate little algebra (the subalgebra of the Poincaré Lie algebra which fixes the given standard 4-momentum). In the present realization, the physical little algebra, say $\mathcal{L}(\overset{\circ}{p})$, is extended by a dual or commutant $\mathbb{L}(\overset{\circ}{p})$ within $U(\mathbb{H}_4)$. That is, there is a certain subalgebra of the full enveloping algebra of the auxiliary Heisenberg algebra, which controls the degeneracy of little algebra unirreps, subject to the constraint (8) which as we have seen, projects the physical state space onto a unitary irreducible representation of the full quaplectic algebra.

In the massive, spacelike and null cases, the little algebras, denoted $\mathcal{L}^{(>0)}, \mathcal{L}^{(<0)}$ and $\mathcal{L}^{(\equiv 0)}$, are the Lie algebras of the orthogonal groups $SO(3)$, $SO(2,1)$ and $SO(3,1)$ respectively. A natural way to find their commutants $\mathbb{L}(\overset{\circ}{p})$ follows if the number space of the auxiliary oscillator modes is identified with an appropriate (metaplectic) unitary representation of the Lie algebra of the eight-dimensional symplectic group $Sp(8,\mathbb{R})$, using the general embeddings based on the group branchings $Sp(2d,\mathbb{R}) \supset Sp(2,\mathbb{R}) \times SO(d)$ or $Sp(2d,\mathbb{R}) \supset Sp(2,\mathbb{R}) \times SO(d-1,1)$ for the appropriate d and real forms.

In the null case for example, $\overset{\circ}{p} = (0,0,0,0)$ and the little algebra is the entire Lorentz group Lie algebra $\mathcal{L}^{(\equiv 0)} = SO(3,1)$; the noncompact embedding with $d = 4$ applies, and the commutant $\mathbb{L}^{(\equiv 0)}$ is denoted $Sp^{(0123)}(2,\mathbb{R})$ reflecting that it is the diagonal sum of the *four* $Sp(2,\mathbb{R})$ oscillator algebras, one for each Cartesian direction in Minkowski space (for details of unitary irreducible representations of $Sp(2,\mathbb{R})$ in relation to oscillator representations, and rules for the reduction of direct products of such representations, see appendix A.2). On the other hand for $\overset{\circ}{p} = (0,0,0,\overset{\circ}{p})$ (the spacelike case), the little algebra $SO(2,1)$, generated by three-dimensional Lorentz transformations in directions 0, 1, 2 in Minkowski space, clearly commutes with both the diagonal $Sp^{(012)}(2,\mathbb{R})$ and $Sp^{(3)}(2,\mathbb{R})$, so $\mathcal{L}^{(<0)} + \mathbb{L}^{(<0)} = SO(2,1) + (Sp^{(012)}(2,\mathbb{R}) + Sp^{(3)}(2,\mathbb{R}))$. Similar considerations show that for $\overset{\circ}{p} = (mc,0,0,0)$ (the massive case), $\mathcal{L}^{(>0)} + \mathbb{L}^{(>0)} = SO(3) + \left(Sp^{(0)}(2,\mathbb{R}) + Sp^{(123)}(2,\mathbb{R})\right)$. Finally in the massless case, the appropriate commutant is to be found within $U(\mathbb{H}_4)$ itself rather than $Sp(8,\mathbb{R})$, and we find $\mathcal{L}^{(=0)} + \mathbb{L}^{(=0)} = \mathcal{E}(2) + \mathbb{E}(2)$, a direct sum of two three-dimensional Euclidean Lie algebras.

Results are listed case-by-case in tables 1 and 2. Table 1 provides the standard reference 4-momenta $\overset{\circ}{p}$, the corresponding little algebra and the commutant in each case. Also given in each case is the explicit combination of diagonal generators of the dual algebra which represent the constraint operator, the difference of number operators $\frac{1}{2}(\widehat{N}_1 + \widehat{N}_2 + \widehat{N}_3 - \widehat{N}_0) + \frac{1}{2}$ (whose Δ-eigenspace, from (8) provides the physical state space). Details of the embeddings within $Sp(8,\mathbb{R})$ and the general $Sp(2d,\mathbb{R}) \supset Sp(2,\mathbb{R}) \times SO(d)$ or $Sp(2d,\mathbb{R}) \supset Sp(2,\mathbb{R}) \times SO(d-1,1)$ constructions are provided in appendix A.3. The identification of the dual algebra for the massless case is described in appendix A.7. In appendices A.3–A.7 details of the unitary irreducible representations of the occurring compact and non-compact Lie algebras in addition to $Sp(2,\mathbb{R})$ can be found (which is treated in appendix A.2). In this notation, table 2 lists for each orbit class, the physical state space(s) occurring, and also their degeneracy, in terms of unitary irreducible representations of the respective dual, commutant algebras (projected onto the Δ-eigenspace as indicated in table 1).

3. Results and discussion

We begin by stating the results of the analysis sketched in the foregoing, as summarized in tables 1 and 2, for each sector of the $p \cdot p$ spectrum, and paying attention to the constraint projection. A brief commentary on these technical results is followed by a discussion of physical aspects and possible extensions of the model.

3.1. Massive states

For fixed $p \cdot p = m^2c^2 > 0$, there is an infinite series of integer-spin particles $\ell = 0, 1, 2, \ldots,$ for each such spin, a degeneracy governed by the tensor product with the indicated projective representation of the dual algebra, which can be enumerated as follows. States of the *reducible* representation $D^-_{-1/4} \oplus D^-_{-3/4}$ are spanned by the eigenvectors of $K_0^{(0)}$, with spectrum $-\frac{1}{4} - \frac{1}{2}k^{(0)}, k^{(0)} = 0, 1, 2, \ldots,$ whereas for fixed ℓ, we have $K_0^{(123)}$ eigenvalues of the form $\frac{1}{2}\ell + \frac{1}{4} + k^{(123)}, k^{(123)} = 0, 1, 2, \ldots$. Thus the constraint reads

$$\left(\tfrac{1}{2}\ell + \tfrac{1}{4} + k^{(123)}\right) + \left(-\tfrac{1}{4} - \tfrac{1}{2}k^{(0)}\right) = \Delta \qquad \text{or} \qquad k^{(123)} - \tfrac{1}{2}k^{(0)} = \Delta - \tfrac{1}{2}\ell \tag{9}$$

so that an infinite tower of recurrences of each ℓ exists (depending on whether $\Delta - \frac{1}{2}\ell$ is integral or half-integral, these are associated with even or odd modes of the $^{(0)}$ oscillator). In fact, regarding the $K_0^{(0)}$ eigenvalue, or just $k^{(0)}$, as being fixed by the choice of $k^{(123)} = 0, 1, 2, \ldots,$ a weight diagram of $[\ell]$ versus $k^{(123)}$ just corresponds to the orbital angular momentum content of a three-dimensional isotropic simple harmonic oscillator system.

3.2. Spacelike (tachyonic) states

For fixed $p \cdot p < 0$, the allowed unirreps of $SO(2, 1)$ (consistent with the constraint) are listed in table 2. The analysis is complicated in this case by three different cases of discrete series representations, and also a continuous series contribution. For example, the auxiliary representation spaces tensored either with the D^+_ℓ *or* D^-_ℓ discrete $SO(2, 1)$ unirreps (for $\ell = 1, 2, \ldots$), are spanned by the eigenstates of $K_0^{(012)}$, with eigenvalues $\frac{1}{2}\ell - \frac{1}{4} + k^{(012)}, k^{(012)} = 0, 1, 2, \ldots$; together with eigenstates of $K_0^{(3)}$, with eigenvalues $\frac{1}{4} + \frac{1}{2}k^{(3)}, k^{(3)} = 0, 1, 2, \ldots,$ and the constraint reads

$$\left(\tfrac{1}{2}\ell - \tfrac{1}{4} + k^{(012)}\right) + \left(\tfrac{1}{4} + \tfrac{1}{2}k^{(3)}\right) = \Delta \quad \text{or} \quad k^{(012)} + \tfrac{1}{2}k^{(3)} = \Delta - \tfrac{1}{2}\ell, \quad \ell = 1, 2, 3, \ldots. \tag{10}$$

The analysis of the remaining cases proceeds similarly. The singlet representation [0] of $SO(2, 1)$ is accompanied by discrete series representations $D^\pm_{-1/4}$ of $Sp^{(012)}(2, \mathbb{R})$ and the constraint becomes

$$D^+_{-1/4}: \quad \left(\tfrac{1}{4} + k^{(012)}\right) + \left(\tfrac{1}{4} + \tfrac{1}{2}k^{(3)}\right) = \Delta, \qquad \text{or} \qquad k^{(012)} + \tfrac{1}{2}k^{(3)} = \Delta - \tfrac{1}{2}, \tag{11a}$$

and

$$D^-_{-1/4}: \quad \left(-\tfrac{1}{4} - k^{(012)}\right) + \left(\tfrac{1}{4} + \tfrac{1}{2}k^{(3)}\right) = \Delta, \qquad \text{or} \qquad -k^{(012)} + \tfrac{1}{2}k^{(3)} = \Delta, \tag{11b}$$

respectively. By contrast, the continuous series representation $D\left(-\frac{1}{2} + \mathrm{i}s\right)$ occurs as a direct integral over s, tensored with the corresponding unirrep $D\left(-\frac{1}{2} + 2\mathrm{i}s\right)$ of $Sp^{(012)}(2, \mathbb{R})$ (which is doubly degenerate). The spectrum of $K_0^{(012)}$ is $\frac{1}{2}m^{(012)} + \frac{1}{4}$ for *integer* $m^{(012)}$, so that together with the $K_0^{(3)}$ eigenstates the constraint condition reads

$$\left(\tfrac{1}{2}m^{(012)} + \tfrac{1}{4}\right) + \left(\tfrac{1}{4} + \tfrac{1}{2}k^{(3)}\right) = \Delta, \qquad \text{or} \qquad \tfrac{1}{2}m^{(012)} + \tfrac{1}{2}k^{(3)} = \Delta - \tfrac{1}{2}. \tag{12}$$

See below for a discussion of the issue of which of these tachyonic states can survive the constraint projection.

3.3. Null states

The analysis proceeds similarly to the discussion of the tachyonic states above. Taking into account the spectrum of $K^{(0123)}$ in the two different cases (discrete series or (doubly degenerate) continuous principal series unirreps of $SO(3,1)$) we have

$$k^{(0123)} = \Delta - \tfrac{1}{2}\ell - \tfrac{1}{2}, \qquad \ell = 0, 1, 2, \ldots \tag{13a}$$

or

$$\tfrac{1}{2}m^{(0123)} = \Delta - \tfrac{1}{2}, \tag{13b}$$

respectively.

3.4. Massless states

Unitary irreducible representations for massless states are of continuous spin type, with a decomposition over continuous $\mathcal{E}(2)$ series unirreps, for Euclidean momentum of length $\pi, 0 < \pi < \infty$ (with arbitrary helicity $0, \pm 1, \pm 2, \ldots$) while table 2 shows that for fixed π, there is (up to re-scaling of the dual momentum label) a *further* continuum of unirreps of the analogous $\mathbb{E}(2)$ dual algebra. Within each such dual unirrep, the constraint simply selects the eigenvalue Δ of the diagonal generator.

As mentioned in the introduction, and as is clear from the above discussion and table 2, the present model shows a complex structure of 'physical' particle states. Firstly, there is no selected mass scale, as the spectrum of $p \cdot p$ is over the whole real line. Even for the massive branch $p \cdot p = m^2c^2 > 0$, there is an infinite number of particles with spins $\ell = 0, 1, 2, \ldots,$ each of which is countably degenerate, so that the spin content as a whole is equivalent to the orbital angular momentum decomposition of a nonrelativistic isotropic three-dimensional simple harmonic oscillator system. As to the $p \cdot p = 0$ branch, unfortunately, in this model, the massless 'particle' states are not of the conventional helicity type, but rather are of continuous spin type, with nonzero Euclidean momentum (and hence arbitrary integer helicity), with each such non-minimal massless unirrep itself being continuously degenerate.

All of the tachyonic states identified constitute an unacceptable violation of causality, but for completeness we have listed them in full as they are unavoidable consequences of our present construction. However, it is apparent from (10)–(12) that, remarkably, it is possible simply by a careful choice of Δ to eliminate some of these states. For example, $\Delta = 0$ or $\Delta = \frac{1}{2}$ removes solutions of (10) and (11*a*), while in (11*b*) and (12) the parity of $k^{(3)}$ and $k^{(012)}$ or $m^{(012)}$ in the respective oscillator spaces (whether even or odd occupation numbers are admissible) is correlated according to whether Δ is integral or half-odd integral. Likewise, the null states, with $\overset{\circ}{p} \equiv 0$, are also not associated with conventional particle states, but similar comments about the possibility of at least partial elimination of some of these states apply—for example from (13*a*), it is evident that again a choice such as $\Delta = 0$ or $\frac{1}{2}$ would serve to remove these unirreps (while leaving the contribution from the continuous series representations unaffected).

Before leaving this overview of technicalities, it is worthwhile pointing out the difference between our survey of the different representation classes, and the original Wigner method applied just to the Poincaré group itself. We stress that the worldline formulation plays a crucial role in *selecting* a certain class of unirreps of the quaplectic group (namely, those (for $\hbar \neq 0$) corresponding to $U(3,1)$ oscillator irreps with Casimir Δ, and higher degree Casimirs given by appropriate algebraic relations [12]). *Given* such a unirrep, our analysis of the massive, massless, spacelike and null Poincaré representations occurring thus constitutes the appropriate group–subgroup branching law. That more general unirreps ('reciprocal-relativistic elementary systems') exist, follows from the induced representation method applied to the quaplectic group itself, as fully described in [6].

We conclude this discussion with some comments on extensions of the present work, which may allow some of these issues arising from the specific results, to be addressed (related comments were made in the concluding remarks of I, but in the context of the results of that earlier approach).

It is striking that, despite the presence of dimensionful constants κ_0 and λ_0 (b/c, and $\hbar$) in the worldline action, there is no selected mass scale. However, from the geometrical point of view, given the close relationship between semi-direct product groups and coset spaces, it is natural to expect that further generalizations of the action (1) can be constructed. Such a modification of the geometry of the 'coordinates' $x^{\mu}(\tau)$, $p_{\nu}(\tau)$, $\theta(\tau)$ may suggest that the continuous spectrum of $p \cdot p$ in the present model is at least an artefact of having a 'flat' target space.

It is clear from, say, (6) and confirmed by table 2, that only integer spin (and helicity) states are possible in the present construction. A natural further step would thus be towards the equivalent of spinning particle or superparticle versions of reciprocally invariant worldline systems. It is conceivable, although not evident how at present, that in this context, the combination of first class constraints in such a super-formulation, might indeed allow a projection onto acceptable, *conventional* types of particles in the physical state space, and exercise the non-minimal and 'unphysical', in a conventional sense, particle states allowed in our present construction.

Finally there is the question of the interpretation of unconventional unirreps such as the continuous spin massless unirreps with nonzero Euclidean momentum, and infinitely many (integer) helicities (see table 2). If they survive in a complete and otherwise consistent model, then they deserve to be taken seriously as a potential component of 'exotic matter'[7]. It is not clear how the present formulation can be extended (via 'second quantization'?) to support interactions. However, if it turns out that 'standard' particle processes are an approximation to a full theory (in an expansion in inverse powers of the mass or energy scale[8] of the Born constant b), then, in particular, couplings of such nonstandard states to conventional matter should be suppressed; they would then tend to be invisible to conventional detectors made of normal particles[9].

We leave such speculations to future work. Born reciprocity is evidently an original and challenging starting point for a theory of generalized elementary particles, which deserves serious study in the same vein as other higher dimensional or string-like models. It shares with them some of the same problems, such as being beset by unphysical particle spectra, and an explosion of modes, although it is unique in that it does not involve any additional 'higher dimensions' (other than the compact θ-coordinate, whose conserved momentum is anyway assumed to be superselected). Of course, it goes without saying that it runs foul of the famous 'no go' theorems [14] which forbid extensions of Poincaré invariance in spacetime in the context of the usual postulates of local quantum field theory, so the unrealistic particle spectrum in the present worldline model is not unanticipated. But, at base, reciprocal relativity is radically different from standard relativistic physics. Not least, it postulates a maximum momentum transfer rate, b, between interacting systems—analogously to the

[7] A recent account of continuous spin massless unirreps and how they arise in supersymmetric and higher spin theories has been given in [27].

[8] For example, equating the 'Born mass' $\sqrt{\hbar b/c^3}$ with the Planck mass $\sqrt{\hbar c/G_N}$ yields $b \cong c^4/G_N \cong 10^{44} N$.

[9] See [15] for comments on such issues as the indeterminacy principle in the context of reciprocity; a local gauged version of quaplectic symmetry has been considered in [28].

maximum displacement rate—the speed of light, c, in relativity. More fundamentally, it overturns conventional understandings of kinematics and dynamics in its very denial of the existence of inertial frames, and its insistence on the ubiquity of interactions, even in the nonrelativistic regime [16].

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Appendix

A.1. Quaplectic algebra $Q(3,1)$

The 25-dimensional quaplectic Lie algebra in $D = 4$ dimensions is generated by $E^{\mu}{}_{\nu}$, $\mu, \nu = 0, 1, 2, 3$ such that (in unitary representations)

$$\left(E^{\mu}{}_{\nu}\right)^{\dagger} = \eta^{\mu\rho}\eta_{\nu\sigma}\left(E^{\sigma}{}_{\rho}\right) \tag{A.1}$$

which generate the real Lie algebra of $U(3,1)$,

$$\left[E^{\mu}{}_{\nu}, E^{\rho}{}_{\sigma}\right] = \delta^{\mu}{}_{\sigma}E^{\rho}{}_{\nu} - \delta_{\nu}{}^{\rho}E^{\mu}{}_{\sigma}, \tag{A.2}$$

together with the complex vector operator Z^{μ} and its conjugate $\overline{Z}^{\mu}$,

$$(Z^{\mu})^{\dagger} = \eta^{\mu\rho}\overline{Z}_{\rho} \equiv \overline{Z}^{\mu} \tag{A.3}$$

which fulfil the Heisenberg algebra (with central generator I)

$$[Z^{\mu}, \overline{Z}^{\nu}] = -\eta^{\mu\nu}I, \tag{A.4}$$

$$[Z^{\mu}, Z^{\nu}] = 0 = [\overline{Z}^{\mu}, \overline{Z}^{\nu}]. \tag{A.5}$$

The E and Z satisfy the commutation relations:

$$\left[E^{\mu}{}_{\nu}, \overline{Z}^{\rho}\right] = \delta_{\nu}{}^{\rho}\overline{Z}^{\mu}, \qquad \left[E^{\mu}{}_{\nu}, Z^{\rho}\right] = -\eta^{\mu\rho}Z_{\nu}. \tag{A.6}$$

In the above, the Lorentz metric $\eta_{4\times4} = \mathrm{diag}(+1, -1, -1, -1)$ is adopted together with standard conventions for raising and lowering indices.

Relativistic position and momentum operators X^{μ}, P^{ν} are defined as the quadrature components of Z^{μ} and $\overline{Z}^{\mu}$, namely[10]

$$\begin{aligned} Z^{\mu} &= \frac{1}{\sqrt{2}}(-X^{\mu} + \mathrm{i}P^{\mu}), \qquad & \overline{Z}^{\mu} &= \frac{1}{\sqrt{2}}(-X^{\mu} - \mathrm{i}P^{\mu}), \\ X^{\mu} &= \frac{1}{\sqrt{2}}(Z^{\mu} + \overline{Z}^{\mu}), \qquad & P^{\mu} &= \frac{\mathrm{i}}{\sqrt{2}}(Z^{\mu} - \overline{Z}^{\mu}), \end{aligned} \tag{A.7}$$

with $\quad [X^{\mu}, P_{\nu}] = \mathrm{i}\delta^{\mu}{}_{\nu}, \quad$ and $\quad [X^{\mu}, X^{\nu}] = 0 = [P^{\mu}, P^{\nu}].$

The structure of the quaplectic algebra is made more transparent in terms of auxiliary generators $\left\{e^{\mu}{}_{\nu}\right\}$ which provide a 'spin–orbit' like decomposition,

$$e^{\mu}{}_{\nu} := E^{\mu}{}_{\nu} - \frac{1}{2I}\{\overline{Z}^{\mu}, Z_{\nu}\}, \qquad E^{\mu}{}_{\nu} = e^{\mu}{}_{\nu} + \frac{1}{2I}\{\overline{Z}^{\mu}, Z_{\nu}\}, \tag{A.8}$$

[10] See footnote 2.

such that the $e^\mu{}_\nu$ satisfy the $U(3,1)$ algebra, but commute with $H(4)$:

$$\left[E^\mu{}_\nu, e^\rho{}_\sigma\right] = \delta^\mu{}_\sigma e^\rho{}_\nu - \delta_\nu{}^\rho e^\mu{}_\sigma, \tag{A.9a}$$

$$\left[e^\mu{}_\nu, e^\rho{}_\sigma\right] = \delta^\mu{}_\sigma e^\rho{}_\nu - \delta_\nu{}^\rho e^\mu{}_\sigma, \tag{A.9b}$$

$$\left[e^\mu{}_\nu, Z^\rho\right] = 0 = \left[e^\mu{}_\nu, \overline{Z}^\rho\right]. \tag{A.9c}$$

As confirmed by the details of Mackey induced representation theory applied to this case, (see [7]), it is clear from this sketch that a generic unitary irreducible representation (unirrep) of the quaplectic group can be associated with the tensor product of a unirrep of $U(3,1)$ (provided by nonzero $e^\mu{}_\nu$), with suitable unirrep(s) of the Weyl–Heisenberg algebra $H(4)$. The latter can of course themselves be identified via induced representations [17].

Finally, in relation to the general quaplectic algebra note that the spin–orbit decomposition (A.8) allows for an easy identification of Casimir operators of Gel'fand type [6]. Define

$$\begin{aligned} &(e^{(n+1)})^\mu{}_\nu = (e^{(n)})^\mu{}_\rho e^\rho{}_\nu, \qquad (e^{(1)})^\mu{}_\nu \equiv e^\mu{}_\nu, \\ &\text{then} \quad C_n = \operatorname{tr}(e^{(n)}) = (e^{(n)})^\mu{}_\mu; \end{aligned} \tag{A.10}$$

tensorially (from (A.9a) these traces are $U(3,1)$ invariants; however from (A.9c) they are trivially also quaplectic Casimirs. Explicitly, we have for example with $U = E^\mu{}_\mu$ (compare (A.8))

$$C_1 = UI - \tfrac{1}{2}(P^\mu P_\mu + X^\mu X_\mu). \tag{A.11}$$

The quaplectic algebra itself can be re-written in a tensor form which identifies its Lorentz (and Poincaré) subalgebras. Identification of the Poincaré subalgebra depends on the choice of Abelian 4-vector operator, which can be either X^μ or P_ν. The generators of the Lorentz group are then $L_{\mu\nu} = \mathrm{i}(E_{\mu\nu} - E_{\nu\mu})$, and the remaining generators (of reciprocal boost transformations) form a symmetric tensor, $M_{\mu\nu} = E_{\mu\nu} + E_{\nu\mu}$, and the commutation relations amongst the homogeneous generators read

$$\begin{aligned} [L_{\kappa\lambda}, L_{\mu\nu}] &= \mathrm{i}(\eta_{\lambda\mu}L_{\kappa\nu} - \eta_{\kappa\mu}L_{\lambda\mu} - \eta_{\lambda\nu}L_{\kappa\mu} + \eta_{\kappa\nu}L_{\lambda\mu}), \\ [L_{\kappa\lambda}, M_{\mu\nu}] &= \mathrm{i}(\eta_{\lambda\mu}M_{\kappa\nu} - \eta_{\kappa\mu}M_{\lambda\mu} + \eta_{\lambda\nu}M_{\kappa\mu} - \eta_{\kappa\nu}M_{\lambda\mu}), \\ [M_{\kappa\lambda}, M_{\mu\nu}] &= (\eta_{\lambda\mu}M_{\kappa\nu} + \eta_{\kappa\mu}M_{\lambda\mu} + \eta_{\lambda\nu}M_{\kappa\mu} + \eta_{\kappa\nu}M_{\lambda\mu}), \end{aligned} \tag{A.12}$$

together with the usual relations expressing the transformation laws expressing the 4-vector nature of X^μ and P_ν, for example,

$$[L_{\kappa\lambda}, P_\mu] = \mathrm{i}(\eta_{\lambda\mu}P_\kappa - \eta_{\kappa\mu}P_\lambda).$$

As mentioned in section 1, in the worldline model there are *two* independent Heisenberg algebras in the construction. For the conserved Noether charges, it is natural to take the usual Schrödinger representation $\mathcal{P}_\mu \to -\mathrm{i}\hbar\partial/\partial x^\mu$ acting on suitable wavefunctions on Minkowski space. Such a representation is of course equivalent to one in which the complex combinations (A.7) are regarded as mode operators for a countable number basis, and it is this representation which is used for the auxiliary Heisenberg algebra carrying the spin degrees of freedom in our model. From (A.4), $\mathbb{Z}_0$ is to be identified with a 'creation' operator, whereas each $\mathbb{Z}_i$ is identified with an 'annihilation' operator (reflecting the sign change in the commutation relations of $\mathbb{Z}^\mu$ and $\overline{\mathbb{Z}}^\nu$ between the temporal and spatial parts):

$$\begin{pmatrix} \mathbb{Z}_0 \\ \mathbb{Z}_1 \\ \mathbb{Z}_2 \\ \mathbb{Z}_3 \end{pmatrix} = \begin{pmatrix} a^\dagger \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad \begin{pmatrix} \overline{\mathbb{Z}}_0 \\ \overline{\mathbb{Z}}_1 \\ \overline{\mathbb{Z}}_2 \\ \overline{\mathbb{Z}}_3 \end{pmatrix} = \begin{pmatrix} a \\ b_1^\dagger \\ b_2^\dagger \\ b_3^\dagger \end{pmatrix}. \tag{A.13}$$

The general state in the number basis is then [18, 19]

$$|n_0, n_1, n_2, n_3\rangle = \frac{a^{\dagger n_0} b_1^{\dagger n_1} b_2^{\dagger n_2} b_3^{\dagger n_3}}{\sqrt{n_0! n_1! n_2! n_3!}} |0, 0, 0, 0\rangle. \tag{A.14}$$

A.2. Unitary irreducible representations of $Sp(2, \mathbb{R})$ and products of discrete series representations

The generators of $Sp(2, \mathbb{R}) \cong SU(1, 1) \cong SO(2, 1)$ are [20–23] $K_\pm$, and K_0, with nonzero commutation relations

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = -2K_0,$$

with $K_+^\dagger = K_-$ and K_0 Hermitian in unitary representations. The eigenvalues of K_0, k say, are for unitary irreducible representations of $SO(2, 1)$, either integer (or half-integer for the spinor case); however for general projective representations of $SU(1, 1), k = E_0 + m$, for integer m and real $0 \leqslant E_0 < 1$. Depending on whether the spectrum of k is unbounded above or unbounded below, or unrestricted, we have the positive or negative discrete, or continuous, series representations. The quadratic Casimir

$$C(Sp(2)) := K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) \equiv K_0^2 - K_0 - K_+K_- \tag{A.15}$$

takes the eigenvalue $j(j+1)$, where

$$k = \begin{cases} -j, -j+1, -j+2, \ldots & \text{for} \quad D_j^+; \quad -j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots; \\ j, j-1, j-2, \ldots, & \text{for} \quad D_j^-; \quad -j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots; \\ 0, \pm 1, \pm 2, \ldots, & \text{for} \quad D(j); \quad j = -\tfrac{1}{2} + \mathrm{i}s, 0 \leqslant s < \infty. \end{cases}$$

Thus for the positive and negative discrete series representations, denoted $D_j^\pm$, j is the minimal (maximal) eigenvalue of k, and $j < 0$. The continuous (principal) series representations (with Casimir $-\frac{1}{4} - s^2$) are denoted $D\left(-\frac{1}{2} + \mathrm{i}s\right)$; we shall not be concerned with the additional, so-called supplementary, continuous series. Finally, the one-dimensional, trivial representation is denoted [0].

Of great importance are the rules for the decomposition of tensor products of discrete series representations, namely the Clebsch–Gordan series [21–23]

$$D_{j_1}^\pm \otimes D_{j_2}^\pm \to \sum_{j=j_1+j_2}^{\infty} \oplus D_j^\pm, \tag{A.16}$$

for the case of two same-type discrete series, while for the tensor product of positive and negative discrete series representations $D_{j_1}^+$ and $D_{j_2}^-$ we have

$$\begin{aligned} &(\text{for } |j_1| > |j_2|) \quad D_{j_1}^+ \otimes D_{j_2}^- \to \left(D_{j_1-j_2}^+ \oplus D_{j_1-j_2+1}^+ \oplus \cdots\right) \oplus \int \mathrm{d}s\, D\left(-\frac{1}{2} + \mathrm{i}s\right), \\ &(\text{for } |j_2| > |j_1|) \quad D_{j_1}^+ \otimes D_{j_2}^- \to \left(D_{j_2-j_1}^- \oplus D_{j_2-j_1+1}^- \oplus \cdots\right) \oplus \int \mathrm{d}s\, D\left(-\frac{1}{2} + \mathrm{i}s\right), \end{aligned} \tag{A.17}$$

where the last, $(m+1)$th, term of the set of discrete series is $D_{j_1-j_2+m}^+$, or $D_{j_2-j_1+m}^-$, such that $-1 \leqslant \pm j_1 \mp j_2 + m < 0$, respectively; for $SO(2, 1)$, this is just -1 or $-\frac{1}{2}$. For the degenerate case $j_1 = j_2 = j$, it turns out that $D_j^+ \otimes D_j^-$ reduces as the integral over the continuous series only, *together with* a copy of the one-dimensional, trivial representation.

The representation structure is easily seen for the case of a single oscillator space, with modes b and $b^\dagger$ say. Defining as usual $b|0\rangle = 0$, and with $[b, b^\dagger] = 1$, it is well known

[21, 23, 24] that the Fock space decomposes with respect to the $Sp(2, \mathbb{R})$ algebra generated by

$$K_+ = \tfrac{1}{2}(b^\dagger)^2, \qquad K_- = \tfrac{1}{2}b^2, \qquad K_0 = \tfrac{1}{2}b^\dagger b + \tfrac{1}{4} \tag{A.18}$$

into the direct sum of two (projective) unirreps of the *positive* discrete series $D^+_{-1/4} \oplus D^+_{-3/4}$. Note that the Casimir eigenvalue is $j(j+1) = -3/16$ for both cases. On the other hand for the *opposite* identifications, for an equivalent oscillator space with modes $a, a^\dagger$, say, we can take instead

$$K_- = \tfrac{1}{2}(a^\dagger)^2, \qquad K_+ = \tfrac{1}{2}a^2, \qquad K_0 = -\tfrac{1}{2}a^\dagger a - \tfrac{1}{4}, \tag{A.19}$$

and the Fock space is now the sum $D^-_{-1/4} \oplus D^-_{-3/4}$ of two (projective) unirreps of the *negative* discrete series, again with Casimir invariant $j(j+1) = -3/16$.

As an example of the application of the rules for tensor product decompositions, and as a simple illustration of the explicit elementary methods deployed below, let us take $\left(D^+_{-1/4} \oplus D^+_{-3/4}\right) \otimes \left(D^+_{-1/4} \oplus D^+_{-3/4}\right)$ corresponding for example to the case of two independent oscillator spaces. We have

$$\begin{aligned}\left(D^+_{-1/4} \otimes D^+_{-1/4}\right) \oplus \left(D^+_{-1/4} \otimes D^+_{-3/4}\right) \oplus \left(D^+_{-3/4} \otimes D^+_{-1/4}\right) \oplus \left(D^+_{-3/4} \otimes D^+_{-3/4}\right)\\ \to D^+_{-1/2} \oplus 2\left(D^+_{-1} \oplus D^+_{-3/2} \oplus D^+_{-2} \oplus D^+_{-5/2} \oplus \cdots\right).\end{aligned} \tag{A.20}$$

A.3. $Sp(2, \mathbb{R}) \times SO(d-1, 1)$ or $Sp(2, \mathbb{R}) \times SO(d)$ duality and Casimir relations

For the orbit classes corresponding to orthogonal group little algebras, the discovery of the correct reductions of physical states for each orbit class into unitary irreducible representations of the little algebras (with multiplicity controlled by the dual, commutant algebras), and hence the correct particle content, can be done by elementary means if the group branching rules are tracked instead, via the decompositions with respect to appropriate combinations of the four independent oscillator algebras (A.13) (for each direction in Minkowski space). A crucial property of these dual algebras and their little algebra partners (see below) is that the respective Casimir invariants are *identical* (up to factors), a fact which greatly facilitates the recognition of unirreps.

The general definition of the dual $Sp(2, \mathbb{R})$ commuting algebras in the chains $Sp(2d, \mathbb{R}) \supset Sp(2, \mathbb{R}) \times SO(d-1, 1)$ or $Sp(2d, \mathbb{R}) \supset Sp(2, \mathbb{R}) \times SO(d)$, for $d = 3$ (for $SO(3)$ or $SO(2, 1)$), $d = 4$ (for $SO(3, 1)$), and also $d = 2$ for intermediate calculations, is as follows. Consider the Heisenberg algebra generators $\mathbb{Z}^a, \overline{\mathbb{Z}}_b$ for $a, b \in I \subseteq \{0, 1, 2, 3\}$ corresponding to the subset of directions in Minkowski space appropriate to the little algebra construction for standard 4-momentum $\overset{\circ}{p}$, with $d = |I|$. Thus the little algebra is spanned by the orthogonal generators $\mathbb{L}_{ab}$ corresponding to $SO(d-r, r)$ with $r = 0$ or 1 depending on *I*. Define the *contracted* combinations

$$K_+ = \tfrac{1}{2}\overline{\mathbb{Z}}^a\overline{\mathbb{Z}}_a, \qquad K_- = \tfrac{1}{2}\mathbb{Z}^a\mathbb{Z}_a, \qquad K_0 = -\tfrac{1}{2}\overline{\mathbb{Z}}^a\mathbb{Z}_a + \tfrac{1}{4}d. \tag{A.21}$$

These generators provide the desired $Sp^{(I)}(2, \mathbb{R})$ Lie algebra and obviously commute with the $\mathbb{L}_{ab}$; these algebras are both subalgebras of $Sp(2d, \mathbb{R})$ which is generated by all quadratic combinations of the $\mathbb{Z}^a, \overline{\mathbb{Z}}_b$ (see for example [15]). Moreover, it is easily shown that the Casimir (A.15), is related to the orthogonal group Casimir $C(SO(d-r, r)) := \tfrac{1}{2}\mathbb{L}^{ab}\mathbb{L}_{ab}$ by

$$4C(Sp^{(I)}(2, \mathbb{R})) = C(SO(d-r, r)) + \tfrac{1}{4}d(d-4) \tag{A.22}$$

(*independently* of the signature of the orthogonal metric).

As an illustration of the method to be elaborated in appendices A.4, A.5, and A.6, consider the implications of (A.22) for the case of $d = 2$ in (A.20) above. For $r = 0$ and Euclidean signature, we have the group branchings

$$Sp(4, \mathbb{R}) \supset SO(2) \times Sp(2, \mathbb{R}); \qquad \text{and} \qquad Sp(4, \mathbb{R}) \supset Sp(2, \mathbb{R}) \times Sp(2, \mathbb{R}) \supset Sp(2, \mathbb{R}), \tag{A.23}$$

whereas in the case $r = 1$, and Minkowski signature we have

$$Sp(4, \mathbb{R}) \supset SO(1, 1) \times Sp(2, \mathbb{R}); \qquad \text{and} \qquad Sp(4, \mathbb{R}) \supset Sp(2, \mathbb{R}) \times Sp(2, \mathbb{R}) \supset Sp(2, \mathbb{R}). \tag{A.24}$$

For each signature type, the final $Sp(2, \mathbb{R})$ group is the same in both branchings; in the second alternative, however, the result of the reduction to the final $Sp(2, \mathbb{R})$ group is precisely (A.20). Thus the dimensions and degeneracies of the occurring $SO(2)$ or $SO(2, 1)$ unirreps are manifested directly in the multiplicities displayed in (A.20).

In the $SO(2)$ case, taking for example $a, b = 1, 2$, the generator is simply $\mathrm{i}L_{12}$, with one-dimensional unirreps $[m]$ (the contragredient being $[-m]$), and Casimir $L_{12}^2 = m^2$. The double multiplicity of $Sp(2, \mathbb{R})$ discrete series thus signals the repetition of a given unirrep, or rather the occurrence of it together with its contragredient. Using (A.22) for $d = 2$,

$$m^2 = C(SO(2)) = 4C(Sp(2, \mathbb{R})) + 1,$$

we finally recover the desired $SO(2) \times Sp(2, \mathbb{R})$ tensor product decomposition in the form

$$\big(D^+_{-1/4} \oplus D^+_{-3/4}\big) \otimes \big(D^+_{-1/4} \oplus D^+_{-3/4}\big) \to \big([0] \otimes D^+_{-1/2}\big) \oplus \left(\sum_{\ell=2}^{\infty} \oplus \big([\ell - 1] \otimes D^+_{-\ell/2}\big) \oplus \big([\,-(\ell - 1)] \otimes D^+_{-\ell/2}\big)\right). \tag{A.25}$$

A.4. Massive states $p \cdot p > 0$

For $\overset{\circ}{p} = mc(1, 0, 0, 0)$ the little algebra $SO(3)$ generators $\mathbb{L}_{ij}, i, j = 1, 2, 3$ commute with $Sp^{(123)}(2, \mathbb{R})$ with branching rules extending those of (A.23), namely

$$Sp(6, \mathbb{R}) \supset SO(3) \times Sp^{(123)}(2, \mathbb{R}); \qquad \text{and} \qquad Sp(6, \mathbb{R}) \supset Sp^{(1)}(2, \mathbb{R}) \times Sp^{(2)}(2, \mathbb{R}) \times Sp^{(3)}(2, \mathbb{R}) \supset Sp^{(12)}(2, \mathbb{R}) \times Sp^{(3)}(2, \mathbb{R}) \supset Sp^{(123)}(2, \mathbb{R}) \tag{A.26}$$

so that the method leading to (A.20) and (A.25) must be extended from the reduction of the two-oscillator tensor product space, to the reduction after carrying out the tensor product with a third oscillator space, and using the relation between Casimir invariants to identify the constituents with respect to $SO(3)$.

The results are as follows. By direct application of (A.16), we have from (A.20),

$$\big(D^+_{-1/4} \oplus D^+_{-3/4}\big)^2 \otimes \big(D^+_{-1/4} \oplus D^+_{-3/4}\big) \cong D^+_{-3/4} \oplus 3D^+_{-5/4} \oplus 5D^+_{-7/4} \oplus 7D^+_{-9/4} \oplus 9D^+_{-11/4} \oplus \cdots, \tag{A.27}$$

and using (A.22) for $d = 3$, with the $SO(3)$ Casimir invariant for unitary irreducible representation $[\ell]$ (of dimension $2\ell + 1$) being in the form

$$\ell(\ell + 1) = C(SO(3)) = 4C(Sp^{(123)}(2, \mathbb{R})) + \tfrac{3}{4},$$

the final reduction to $SO(3) \times Sp^{(123)}(2, \mathbb{R})$ reads

$$\big(D^+_{-1/4} \oplus D^+_{-3/4}\big)^3 \to \big([0] \otimes D^+_{-3/4}\big) \oplus \big([1] \otimes D^+_{-5/4}\big) \oplus \big([2] \otimes D^+_{-7/4}\big) \oplus \big([3] \otimes D^+_{-9/4}\big) \oplus \cdots \cong \sum_{\ell=0}^{\infty} \oplus \big([\ell] \otimes D^+_{-\ell/2-3/4}\big). \tag{A.28}$$

A.5. Tachyonic states $p \cdot p < 0$

For $\overset{\circ}{p} = \overset{\circ}{p}_z(0,0,0,1)$ the little algebra $SO(2,1)$ generators $\mathbb{L}_{ab}$, $a, b = 0, 1, 2$ commute with $Sp^{(012)}(2,\mathbb{R})$ with branching rules extending those of (A.23), namely

$$Sp(6,\mathbb{R}) \supset SO(2,1) \times Sp^{(012)}(2,\mathbb{R});$$

$$\text{and} \quad Sp(6,\mathbb{R}) \supset Sp^{(0)}(2,\mathbb{R}) \times Sp^{(12)}(4,\mathbb{R}) \supset Sp^{(0)}(2,\mathbb{R}) \times Sp^{(12)}(2,\mathbb{R}) \times SO(2) \supset SO(2) \times Sp^{(012)}(2,\mathbb{R}). \tag{A.29}$$

In this case the final two-oscillator space decomposition (A.25) must again be extended by carrying out the tensor product with a third oscillator space. However, in contrast to the $SO(3)$ decomposition, in this case from (A.13), (A.21), the realization of the $Sp^{(0)}(2,\mathbb{R})$ algebra requires the identification of the oscillator Fock space with the direct sum $D^-_{-1/4} \oplus D^-_{-3/4}$ of *negative* discrete series representations, and the rules (A.17) must be used. The unirreps of $SO(2)$ occurring can however be used along with knowledge of the spectrum of the diagonal generator $\mathrm{i}\mathbb{L}_{12}$ in unirreps of $SO(2,1) \cong SU(1,1)$, and the relation between Casimir invariants, again

$$C(SO(2,1)) = 4C(Sp^{(012)}(2,\mathbb{R})) + \tfrac{3}{4},$$

to identify the constituents with respect to $SO(2,1)$. The result for the $SO(2) \times Sp^{(012)}(2,\mathbb{R})$ decomposition is

$$\begin{aligned}&\left(D^+_{-1/4} \oplus D^+_{-3/4}\right)^2 \otimes \left(D^-_{-1/4} \oplus D^-_{-3/4}\right) \to \left([0] \otimes (D^+_{-1/4} \oplus D^-_{-1/4})\right) \oplus \\ &\oplus\left(([+1] \oplus [+2] \oplus [+3] \oplus \cdots) \otimes D^+_{-1/4}\right) \oplus \left(([-1] \oplus [-2] \oplus [-3] \oplus \cdots) \otimes D^+_{-1/4}\right) \\ &\oplus\left(([+2] \oplus [+3] \oplus [+4] \oplus \cdots) \otimes D^+_{-3/4}\right) \oplus \left(([-4] \oplus [-3] \oplus [-4] \oplus \cdots) \otimes D^+_{-3/4}\right) \\ &\oplus 2(\cdots [-2] \oplus [-1] \oplus [0] \oplus [+1] \oplus [+2] \oplus \cdots) \otimes \int D\left(-\frac{1}{2} + \mathrm{i}s'\right) \mathrm{d}s'\end{aligned} \tag{A.30}$$

from which we finally recover the $SO(2,1) \times Sp^{(012)}(2,\mathbb{R})$ decomposition

$$\begin{aligned}&\left(D^+_{-1/4} \oplus D^+_{-3/4}\right)^2 \otimes \left(D^-_{-1/4} \oplus D^-_{-3/4}\right) \\ &\quad \to \left([0] \otimes (D^+_{-\frac{1}{4}} \oplus D^-_{-\frac{1}{4}})\right) \oplus \left(\sum_{\ell=1}^{\infty} \left(D^+_{-\ell} \otimes D^+_{-\frac{1}{2}\ell+\frac{1}{4}}\right) \oplus \left(D^-_{-\ell} \otimes D^+_{-\frac{1}{2}\ell+\frac{1}{4}}\right)\right) \\ &\quad \oplus 2\left(\int D\left(-\frac{1}{2} + \mathrm{i}s\right) \otimes D\left(-\frac{1}{2} + 2\mathrm{i}s\right) \mathrm{d}s\right).\end{aligned} \tag{A.31}$$

A.6. Null states $p \equiv 0$

The relevant branching chain is

$$Sp(8,\mathbb{R}) \supset Sp^{(0123)}(2,\mathbb{R}) \times SO(3,1) \tag{A.32}$$

so that the degeneracy of null 'particle' states is controlled by the decomposition with respect to $Sp^{(0123)}(2,\mathbb{R})$. The structure of this Fock space (comprising the a mode together with all three of the b modes) with respect to the total $Sp^{(0123)}(2,\mathbb{R})$ is thus that of the reduction of the tensor product decomposition $Sp^{(0)}(2) \times SO(3) \times Sp^{(123)}(2)$,

$$\left(D^-_{-\frac{1}{4}} \oplus D^-_{-\frac{3}{4}}\right) \otimes \left(\sum_{\ell=0}^{\infty} \oplus [\ell] \otimes D^+_{-\ell-\frac{3}{4}}\right), \tag{A.33}$$

resulting in an $SO(3) \times Sp^{(0123)}(2)$ decomposition from which can be inferred the $SO(3,1) \times Sp^{(0123)}(2)$ branching, knowing the structure of unirreps of $SO(3,1)$ in an $SO(3)$ basis. At the $SO(3) \times Sp^{(0123)}(2)$ level this reads

$$\sum_{\ell=0}^{\infty}\oplus[\ell] \otimes \left(D^{+}_{-\frac{1}{2}\ell-\frac{1}{2}} \oplus D^{+}_{-\frac{1}{2}\ell+\frac{1}{2}} \oplus \cdots \oplus D^{+}_{-\frac{1}{2}} \oplus 2\int \mathrm{d}s\, D\left(-\frac{1}{2}+\mathrm{i}s\right)\right) \tag{A.34}$$

which can be rearranged as

$$2\left(\int \mathrm{d}s\, D\left(-\frac{1}{2}+\mathrm{i}s\right) \otimes \sum_{\ell=0}^{\infty}\oplus\,[\ell]\right) \oplus \left(D^{+}_{-\frac{1}{2}\ell-\frac{1}{2}} \otimes \sum_{\ell'=\ell}^{\infty}\oplus[\ell']\right). \tag{A.35}$$

Unirreps of $SO(3,1)$ are denoted $D[k_0, c]$ according to the parametrization of the eigenvalue $k_0^2 + c^2 - 1$ of the aforementioned $C(SO(3,1))$ Casimir for this case, together with the eigenvalue $\mathrm{i}k_0c$ of the second quadratic Casimir $C'(SO(3,1) = \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}L_{\mu\nu}L_{\rho\sigma}$ (see [25]). In either case $2|k_0|$ is integral, and the angular momentum ($SO(3)$) spectrum is $\ell = |k_0|, |k_0|+1, \ldots$. For Lorentz transformations associated with orbital angular momentum (as is the case for $\mathbb{L}_{\mu\nu}$ in the auxiliary space) C' vanishes, and the relevant representations are in the principal series with either $k_0 = 0$, namely $D[0, -2\mathrm{i}s]$, or $c = 0$, namely $D[\ell, 0]$; the C eigenvalues being $-1-4s^2$ and $\ell^2 - 1$, respectively. Given the interrelationship (A.22) for this case,

$$C(SO(3,1)) = 4C(Sp^{(0123)}(2,\mathbb{R})),$$

it is easily checked therefore that the reduction with respect to $Sp^{=0}(2) \times SO(3,1)$ is simply

$$2\left(\int \mathrm{d}s D\left(-\frac{1}{2}+\mathrm{i}s\right) \otimes D[0,-2\mathrm{i}s]\right) \oplus \left(\sum_{\ell=0}^{\infty}\oplus D^{+}_{-\frac{1}{2}\ell-\frac{1}{2}} \otimes D[\ell,0]\right). \tag{A.36}$$

A.7. $\mathcal{E}(2) \times \mathbb{E}(2)$ duality and massless states $p \cdot p = 0$

In the massless case the little algebra $\mathcal{L}^{(=0)} = \mathcal{E}(2)$ involves noncompact generators such as $\mathbb{L}_{01} + \mathbb{L}_{03}$ whose diagonalization requires some analytical technicalities [26]. This algebra does not admit an $Sp(2,\mathbb{R})$ type commutant, but instead has a dual $\mathbb{E}(2)$ of the same Euclidean type either within $Q(3,1)$ or $Sp(8,\mathbb{R})$.

In a standard basis the Lie algebra $E(2)$ has generators $T_\pm$ and M (the compact $SO(2)$ generator), with commutation relations

$$[M, T_\pm] = \pm T_\pm, \qquad [T_+, T_-] = 0, \tag{A.37}$$

and Casimir invariant $C(E(2)) = T_+T_- = T_1^2 + T_2^2$ where $T_\pm = T_1 \pm \mathrm{i}T_2$ in terms of Hermitian translation generators in Cartesian coordinates. For continuous series unirreps $D(\pi)$, the Casimir eigenvalue is π^2, $0 < \pi < \infty$, and with respect to a standard basis with M diagonal (and integer, or half-odd integer for spin representations), matrix elements

$$\begin{aligned} &T_\pm = \pi|\pi, m \pm 1\rangle, \qquad M|\pi, m\rangle = m|\pi, m\rangle, \\ &\text{and} \qquad \langle\pi', m'|\pi, m\rangle = \delta(\pi' - \pi)\delta_{m'm}. \end{aligned} \tag{A.38}$$

On the other hand, the discrete series representations $[m]$ are one-dimensional helicity states $M|m\rangle = m|m\rangle$ and vanishing Casimir.

A useful step to understanding the full decomposition into massless states in the present, quaplectic realization, is to consider the case of a two-dimensional oscillator system (see (A.13) for index set $I = \{1, 2\}$). For this system $M = \mathrm{i}L_{12}$ as usual, and the translations $T_\pm$ are nothing but the complex momentum combinations $P_\pm = P_1 \pm \mathrm{i}P_2$. Adopting the appropriate raising and lowering combinations for circular oscillators, $b_\pm = \frac{1}{\sqrt{2}}(b_1 \mp b_2)$ we have

$$M = b_+^\dagger b_+ - b_-^\dagger b_-, \qquad P_\pm = b_\pm - b_\mp^\dagger,$$
$$P_+P_- = 1 + b_+^\dagger b_+ + b_-^\dagger b_- - b_+^\dagger b_-^\dagger - b_+ b_-. \tag{A.39}$$

In spherical polar coordinates (r, ϕ), we have $M \rightarrow -\mathrm{i}\partial/\partial\phi$, and $P_\pm \rightarrow -\mathrm{i}\,\mathrm{e}^{\pm\mathrm{i}\phi}(\partial/\partial r \pm (\mathrm{i}/r)\partial/\partial\phi)$, and diagonalization leads to the well-known eigenfunctions $\langle r, \phi|\pi, m\rangle \cong J_m(\pi r)\exp(im\phi)$ so that the oscillator space decomposes into a direct integral over $0 < \pi < \infty$ of $D(\pi)$.

In the present quaplectic realization, the algebraic structure is best seen via a complex tetrad basis relative to the reference lightlike 4-momentum, namely

$$\overset{\circ}{p} = (1, 0, 0, 1), \qquad \overset{\circ}{p}{}' = (1, 0, 0, -1), \qquad u^+ = \tfrac{1}{\sqrt{2}}(0, 1, \mathrm{i}, 0) = \overline{u}^-.$$

Then defining for 4-vectors $v, w, \ldots \mathbb{Z}(v) = \mathbb{Z}^\mu v_\mu$, $\mathbb{L}(v, w) = v^\mu w^\nu \mathbb{L}_{\mu\nu} = -\mathbb{L}(w, v)$, and so on, we have the generators:

$$\mathcal{E}(2): \quad T_\pm = \mathbb{L}(\overset{\circ}{p}, u^\pm), \qquad M = \mathbb{L}(u^+, u^-);$$
$$\mathbb{E}(2): \quad \mathbb{T}_+ = \overline{\mathbb{Z}}(\overset{\circ}{p}), \qquad \mathbb{T}_- = \mathbb{Z}(\overset{\circ}{p}), \qquad \mathbb{M} = \overline{\mathbb{Z}} \cdot \mathbb{Z}; \tag{A.40}$$

note that an alternative commutant can indeed be found within the full $Sp(8, \mathbb{R})$ symplectic algebra, by defining simply $\mathbb{T}'_\pm = \mathbb{T}^2_\pm$, $\mathbb{M}' = 2\mathbb{M}$.

Explicit forms of the generators can be constructed by referring to (A.13); it is convenient to re-label $b_3 \equiv c$, $b_3^\dagger \equiv c^\dagger$ along with $a, a^\dagger$ in order to distinguish the longitudinal 0, 3 ('$\|$') and transverse 1, 2 or $+, -$ directions. We have

$$\mathbb{T}_+ = a^\dagger - c, \quad \mathbb{T}_- = a - c^\dagger,$$
$$\mathbb{M} = (a^\dagger a - c^\dagger c) - \left(b_+^\dagger b_+ + b_-^\dagger b_-\right); \tag{A.41}$$
$$\mathbb{T}_+\mathbb{T}_+ = 1 + a^\dagger a + c^\dagger c - a^\dagger c^\dagger - ac;$$

$$\text{whereas} \quad T_\pm = (a^\dagger - c)b_\pm^\dagger - (a - c^\dagger)b_\mp \equiv \mathbb{T}_+ b_\pm^\dagger - \mathbb{T}_- b_\mp,$$
$$M = b_+^\dagger b_+ - b_-^\dagger b_-,$$
$$\text{and finally} \quad T_+T_- = \left(\mathbb{T}_-^2 b_+^\dagger b_-^\dagger + \mathbb{T}_+^2 b_+ b_-\right) - \mathbb{T}_+\mathbb{T}_-\left(1 + b_+^\dagger b_+ + b_-^\dagger b_-\right). \tag{A.42}$$

There is an obvious equivalence between (A.39), (A.41) which can be used to reduce the 0, 3 oscillator space to a direct integral of continuous series $\mathbb{E}(2)$ unirreps $D(\pi_\|)$. Moreover, from (A.42) on such states $|\psi; \pi_\|, m_\|\rangle \equiv |\psi\rangle \otimes |\pi_\|, m_\|\rangle$, where $|\psi\rangle$ is an arbitrary state of the 1, 2 oscillator space, we have evidently

$$T_\pm|\psi; \pi_\|, m_\|\rangle = \pi_\|\left(b_\pm^\dagger - b_\mp\right)|\psi; \pi_\|, m_\|\rangle;$$
$$M|\psi; \pi_\|, m_\|\rangle = \left(b_+^\dagger b_+ - b_-^\dagger b_-\right)|\psi; \pi_\|, m_\|\rangle \tag{A.43}$$

so that (if necessary by rescaling generators $T_\pm \rightarrow T_\pm/\pi_\|$), essentially the *same* diagonalization renders a direct integral of states of the form

$$|\pi\pi_\|, m; \pi_\|, m_\|\rangle, \tag{A.44}$$

with transverse quantum number (the helicity, the difference between the positive and negative circular oscillator mode numbers) m, and auxiliary mode number $m_\|$. The final projection onto physical states obeying the constraint is thus trivial in the present case: we require simply $\Delta = m_\|$.

References

[1] Govaerts J, Jarvis P D, Morgan S O and Low S G 2007 World-line quantisation of a reciprocally invariant system *J. Phys. A: Math. Theor.* **40** 12095–111

[2] Born M 1949 Elementary particles and the principle of reciprocity *Nature* **163** 207–8

[3] Born M 1949 Reciprocity theory of elementary particles *Rev. Mod. Phys.* **21** 463–73

[4] Green H S 1949 Quantized field theories and the principle of reciprocity *Nature* **163** 208–9

[5] Green H S 1966 Theory of reciprocity, broken $SU(3)$ symmetry, and strong interactions *Proc. Int. Conf. on Elementary Particles (Kyoto, 1965) Prog. Theory Phys.* 159
See also Bracken A J 1970 *PhD Thesis* University of Adelaide

[6] Low S G 2002 Representations of the Canonical group (the semidirect product of the unitary and Weyl–Heisenberg groups), acting as a dynamical group on noncommutative extended phase space *J. Phys. A: Math. Gen.* **35** 5711–29

[7] Low S G 2007 Reciprocal relativity of noninertial frames: quantum mechanics *J. Phys. A: Math. Theor.* **40** 3999–4016

[8] Low S G 2006 Reciprocal relativity of noninertial frames and the quaplectic group *Found. Phys.* **36** 1036–69

[9] See for example Govaerts J 1991 *Hamiltonian Quantisation and Constrained Dynamics* (Leuven: Leuven University Press) (and references therein)

[10] Govaerts J 2004 The cosmological constant of one-dimensional matter coupled quantum gravity is quantised *Proc. 3rd Int. Workshop on Contemporary Problems in Mathematical Physics (2003), Cotonou (Republic of Benin)* ed J Govaerts, M N Hounkonnou and A Z Msezane (Singapore: World Scientific) pp 244–72 (arXiv:hep-th/0408022)

[11] Wigner E P 1939 On unitary representations of the inhomogeneous Lorentz group' and elementary particle classification *Ann. Math.* **40** 149

[12] Shamaly A 1974 Invariants of Born reciprocity theory *J. Math. Phys.* **15** 1178–80

[13] See for instance, Govaerts J and Villanueva V M 2000 Topology classes of flat $U(1)$ bundles and diffeomorphic covariant representations of the Heisenberg algebra *Int. J. Mod. Phys.* A **15** 4903 (and references therein)

[14] O'Raifeartaigh L 1965 Internal symmetry and Lorentz invariance *Phys. Rev. Lett.* **14** 332
O'Raifeartaigh L 1965 Mass differences and lie algebras of finite order *Phys. Rev. Lett.* **14** 575
O'Raifeartaigh L 1965 Lorentz invariance and internal symmetry *Phys. Rev.* **139** B1052
Fuchs G 1968 About O'Raifeartaigh's theorem *Ann. de l'Inst. Henri Poincaré* **9** 7–16

[15] Jarvis P D and Morgan S O 2006 Born reciprocity and the granularity of spacetime *Found. Phys. Lett.* **19** 501–17

[16] Low S G 2008 Hamiltonian relativity group for noninertial states in quantum mechanics *J. Phys. A: Math. Theor.* **30** 304034

[17] Wolf J A 1975 Representations of certain semidirect product groups *J. Funct. Anal.* **19** 339–72

[18] Dirac P A M 1945 Unitary representations of the Lorentz group *Proc. R. Soc. Lond.* A **183** 285–95

[19] Kim Y S and Noz M E 1991 *Phase Space Picture of Quantum Mechanics: Group Theoretical Approach* (Singapore: World Scientific)

[20] Bargmann V 1947 Irreducible unitary representations of the Lorentz group *Ann. Math.* **48** 568–640

[21] Holman W J and Biedenharn L C 1966 Complex angular momenta and the groups $SU(1,1)$ and $SU(2)$ *Ann. Phys.* **39** 1–42
Holman W J and Biedenharn L C 1968 A general study of the Wigner coefficients of $SU(1,1)$ *Ann. Phys.* **47** 205–31

[22] Wang K-H 1970 Clebsch–Gordan series and the Clebsch–Gordan coefficients of $O(2,1)$ and $SU(1,1)$ *J. Math. Phys.* **11** 2077–95

[23] Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)

[24] Gerry C C 2004 On the Clebsch–Gordan problem for $SU(1,1)$: coupling non-standard representations *J. Math. Phys.* **45** 1180–90

[25] Gel'fand I, Minlos R and Shapiro Y 1963 *Representations of the Rotation and Lorentz Groups* (New York: Academic)

[26] Lindblad G and Nagel B 1970 Continuous bases for unitary irreducible representations of $SU(1,1)$ *Ann. de l'Inst. Henri Poincaré* **13** 27–56

[27] Brink L, Khan A M, Ramond P and Xiong X Z 2002 Continuous spin representations of the Poincaré and super-Poincaré groups *J. Math. Phys.* **43** 6279–95

[28] Castro C 2008 Born's reciprocal general relativity theory and complex nonabelian gravity as gauge theory of the quaplectic group : a novel path to quantum gravity *Int. J. Mod. Phys.* A **23** 1487–506